KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

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„SIGNAL PROCESSING”

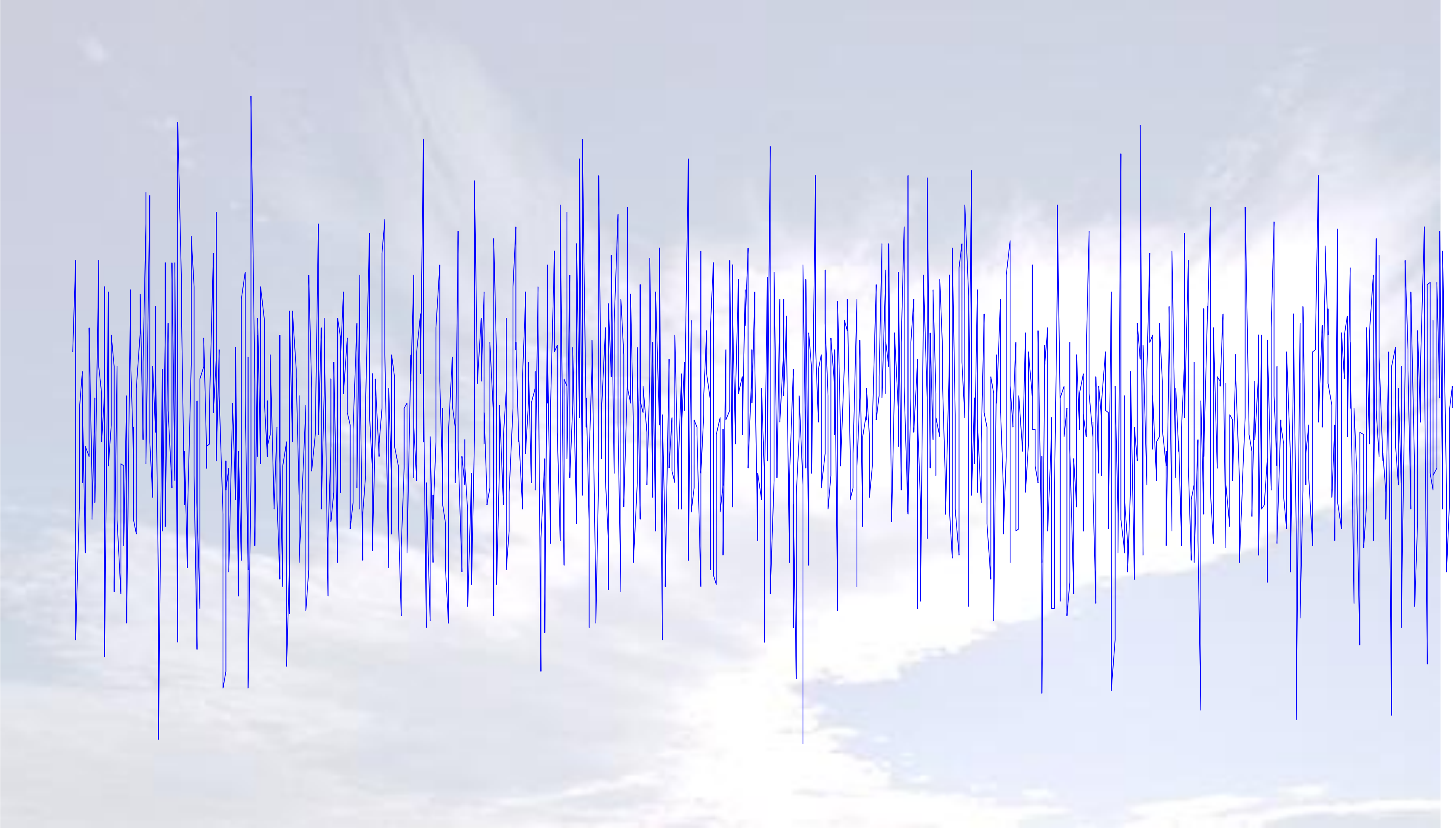
Prezentacja multimedialna współfinansowana przez Unię Europejską w ramach Europejskiego Funduszu Społecznego w projekcie pt.

„Innowacyjna dydaktyka bez ograniczeń - zintegrowany rozwój Politechniki Łódzkiej - zarządzanie Uczelnią, nowoczesna oferta edukacyjna i wzmacniania zdolności do zatrudniania osób niepełnosprawnych”

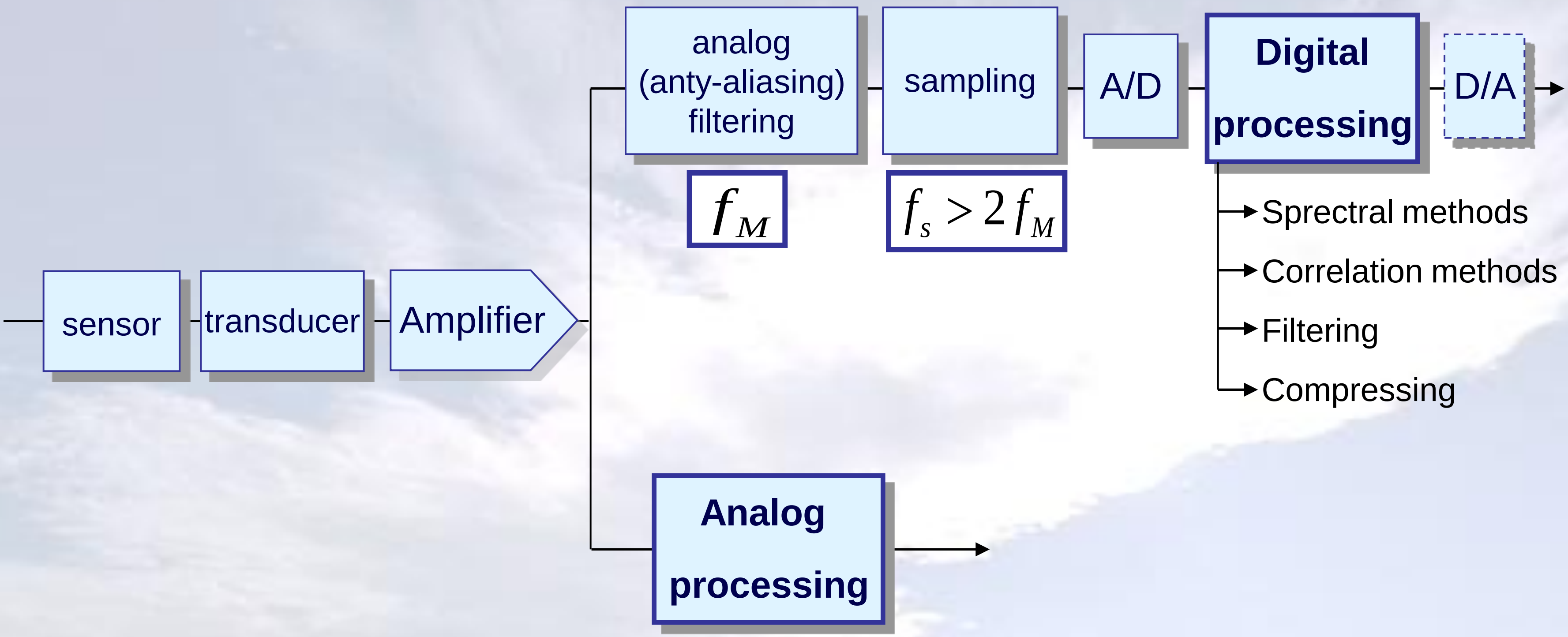
Politechnika Łódzka

Politechnika Łódzka, ul. Żeromskiego 116, 90-924 Łódź, tel. (042) 631 28 83
www.kapitalludzki.p.lodz.pl

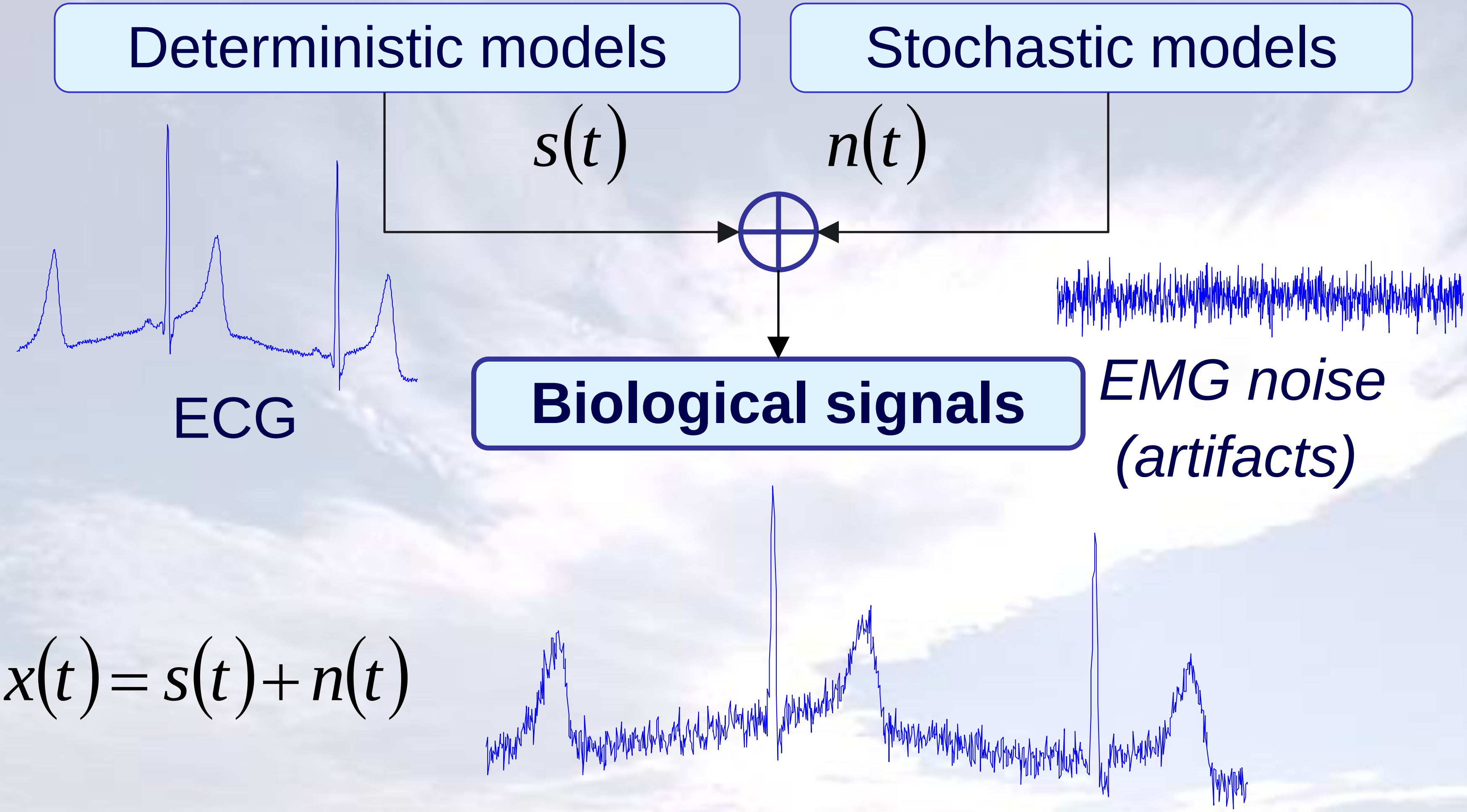
Analysis of Random signals



Signal processing



Biological signals



$$x(t) = s(t) + n(t)$$

Signal models

Signals

- Deterministic
  - Periodic
    - $x(t) = x(t + nT)$
    - Harmonic
    - Compound
  - Non-periodic
    - Quasi-periodic
    - Transient
    - others
- Stochastic
  - **Stationary**
    - Ergodic
    - Non-ergodic
  - Non-stationary

Deterministic chaos models

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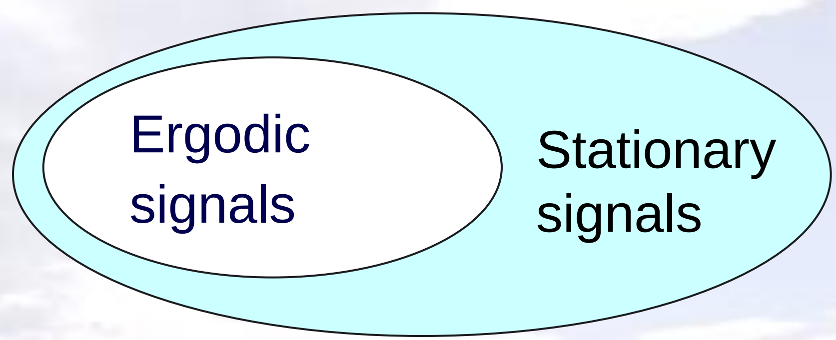
Stationary and ergodic signals (processes)

Stationary signals

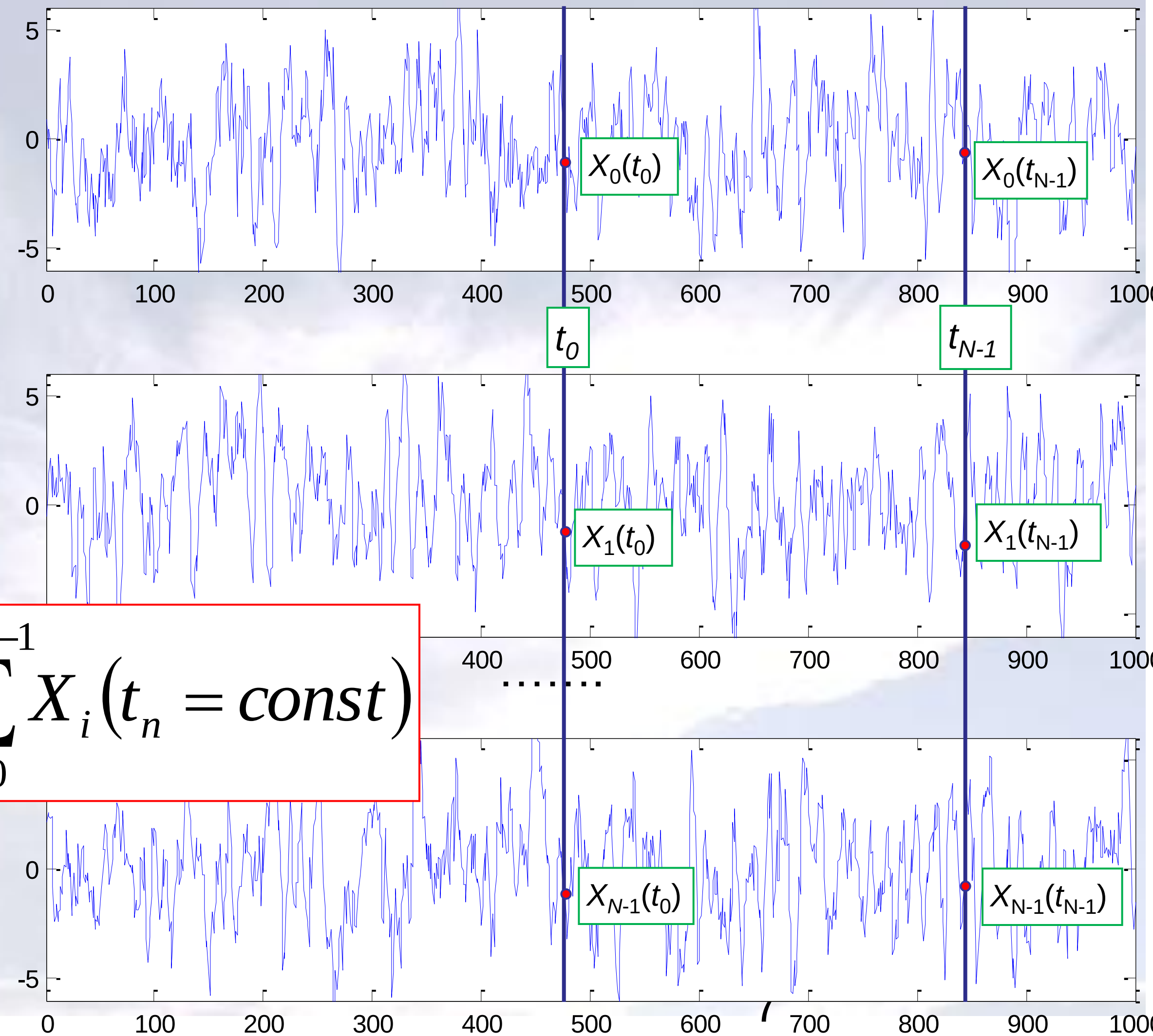
Statistical moments calculated over time (e.g. mean, variance) of a stochastic signal do not change with time.

Ergodic signals

Statistical moments of a signal (e.g. mean, variance) calculated over time and calculated over realizations (ensemble moments) are correspondingly equal.



Stationary and ergodic signals (processes)



A stochatic signal is ergodic if:

$$\frac{1}{N}\sum_{t=0}^{t=N-1} X_{i=const}(t) = \frac{1}{N}\sum_{i=0}^{i=N-1} X_i(t_n = const)$$

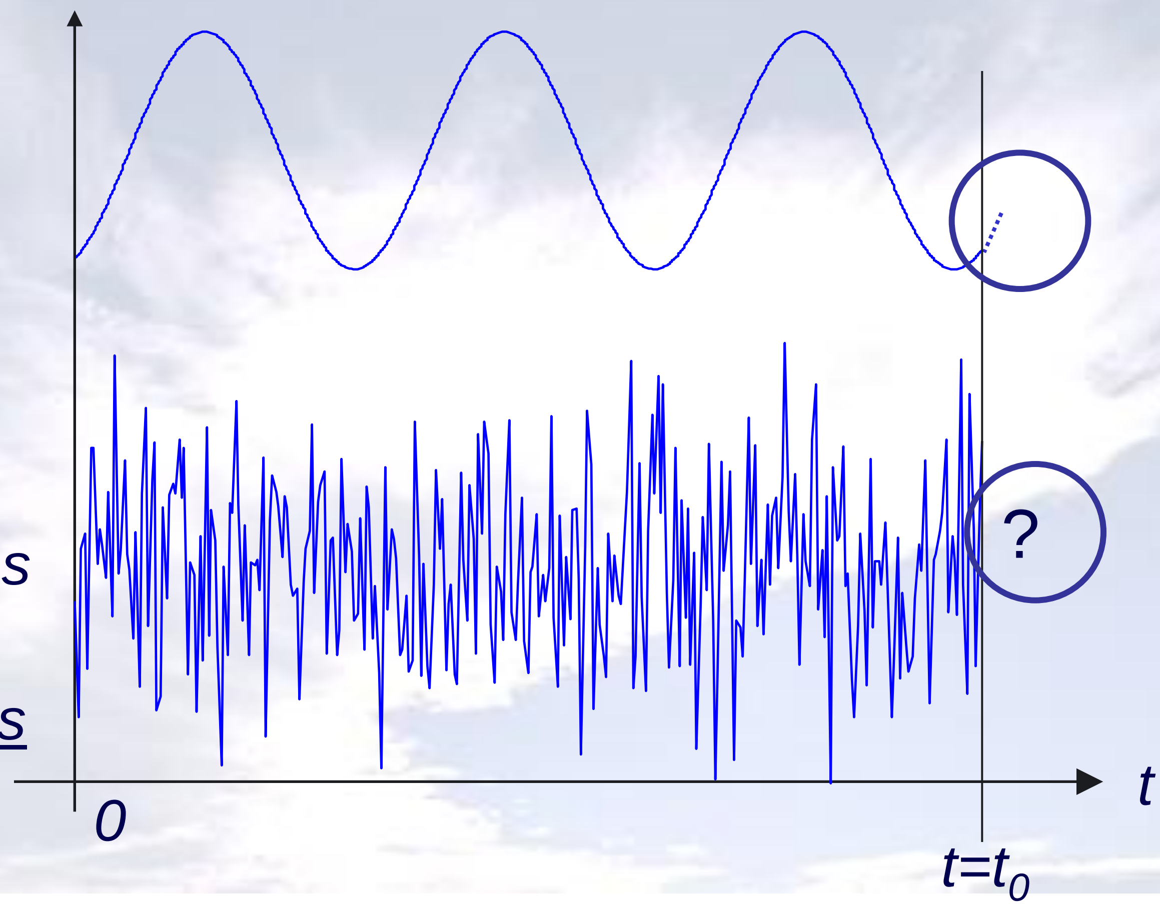
Random vs stochastic signal

deterministic

(samples may be predicted with high accuracy)

random

(unpredictable values of samples, <u>only statistical parameters may be established</u>)



Random signal (parameters)

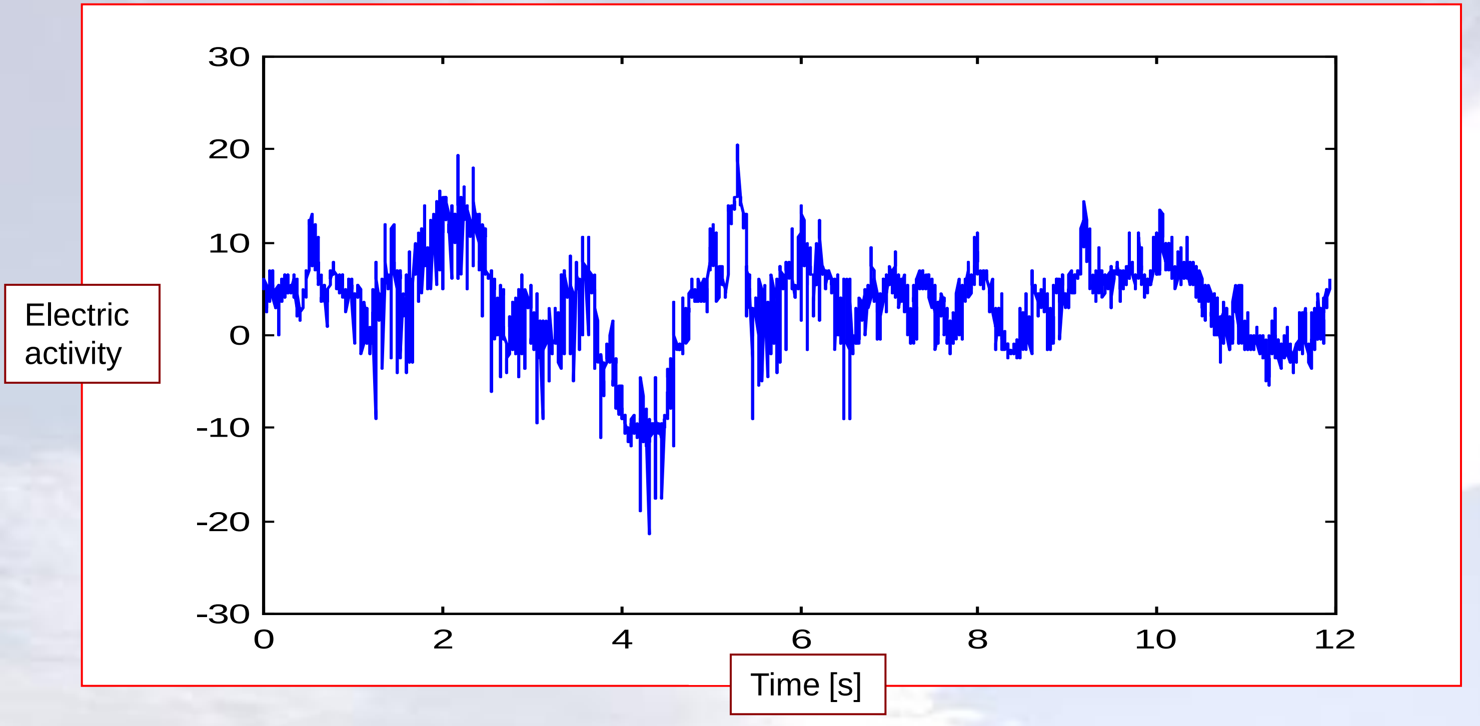
- exact values unpredictable
- can be interpreted as random variable
  (and expressed in terms of the following parameters):
    - expected value (**mean**)
    - **variance** and **standard deviation**
    - **autocorrelation** function

For stationary random signals these parameters are time invariant.

Example – EEG signal

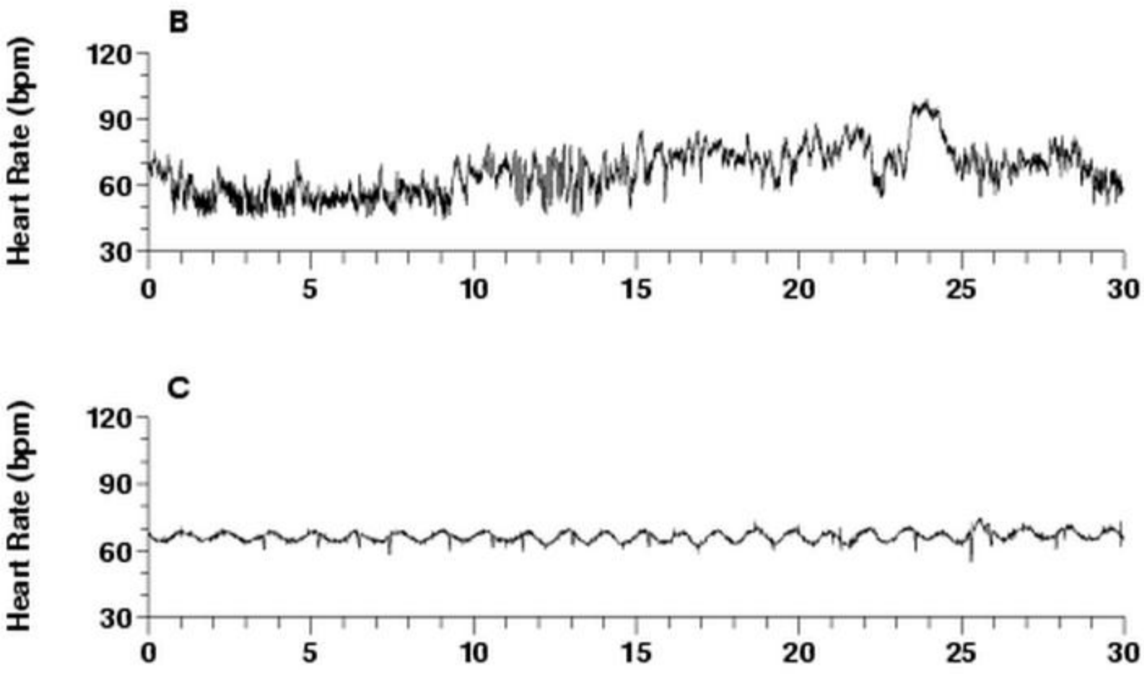
Stochastic model may be applied to electroencephalografic signal (EEG)

EEG is a result of electric activity of large amount * of neurons.



*) It is being proven that neural perception is based on synchronization of specialised populations of neurons

Heart rate (HR) in health and disease

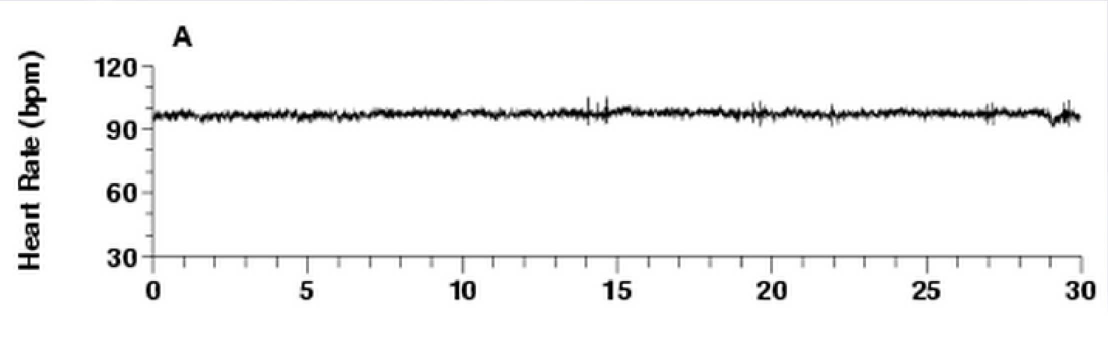


Normal sinus rhythm (nonstationarity, „fractality")

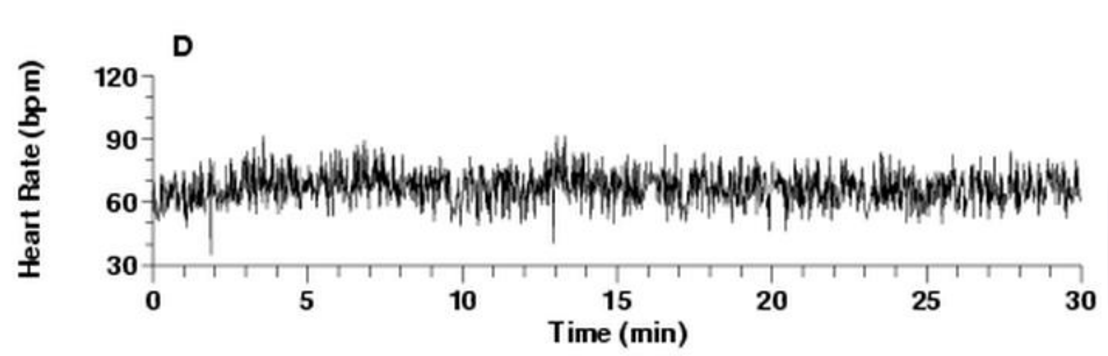
HR in severe congestive heart failure (periodic oscillations presents (~1/min) due to Cheyene-Stokes breathing)

www.physionet.org

Heart rate (HR) in health and disease



HR in severe congestive heart failure – excessive regularity of the heart rate



HR at cardiac arrhythmia, (atrial fibrillation), uncorrelated randomness present

www.physionet.org

Random signal – expected value

Continuous signal:

$$E\{x(t)\} = \int_{-\infty}^{+\infty} x p(x) dx = \bar{x}$$

```
#Python
#see also random?
x=random.randn(1000)
expected=x.mean()
```

Discrete signal:

$$E\{x(n)\} = \sum_{n=-N}^{n=N} x(n) p(n) = \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n) = \bar{x}$$

successive samples are equally probable

Random signal - variance

Variance- *square mean of deviations from the expected value*

Continuous signal:

$$\sigma^2 = E\left\{\left[x(t) - \bar{x}\right]^2\right\} = \int_{-\infty}^{+\infty} \left[x(t) - \bar{x}\right]^2 p(x)\,dx$$

```
#Python
x=random.randn(1000)
std_dev=x.std()
Variance=x.var()
```

Disrete signal:

$$\sigma^2 = \frac{1}{2N+1} \sum_{n=-N}^{n=N} \left[x(n) - \bar{x}\right]^2 = \frac{1}{2N+1} \sum_{n=-N}^{n=N} x^2(n) - \bar{x}^2$$

Random signal – autocorrelation function

Continuous signal:

$$R(\tau)=\int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

Discrete signal:

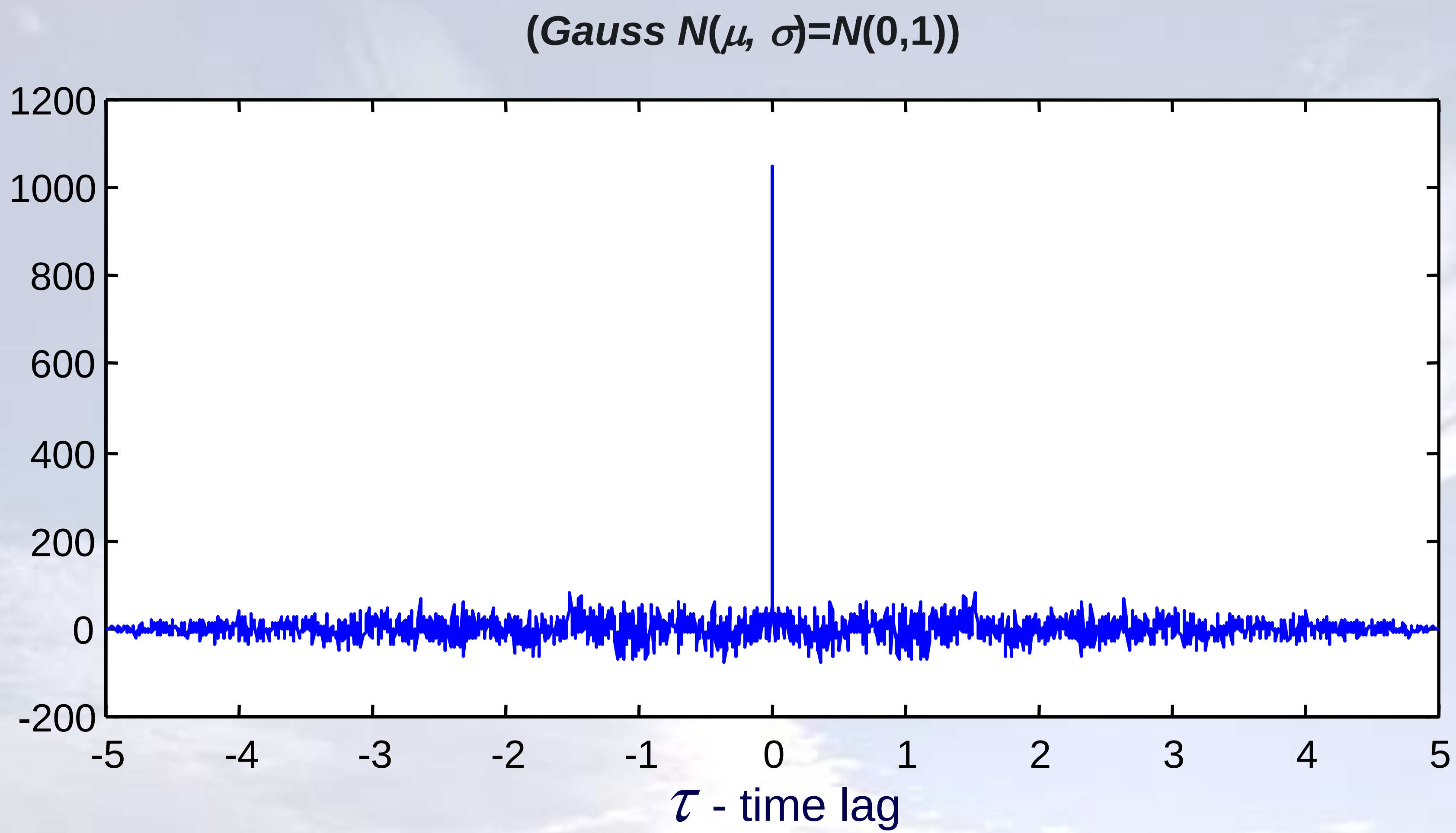
$$R(m)=\sum_{n=-\infty}^{n=\infty} x(n)x(n+m)$$

```
#Python
x=random.randn(1000)
R=correlate(x,x,mode='full')
plot(R)

#better way, remove DC
x=x-x.mean()
R=correlate(x,x,mode='full')
plot(R)
```

Don’t forget to remove the mean from the signal!

Autocorrelation function of a random signal



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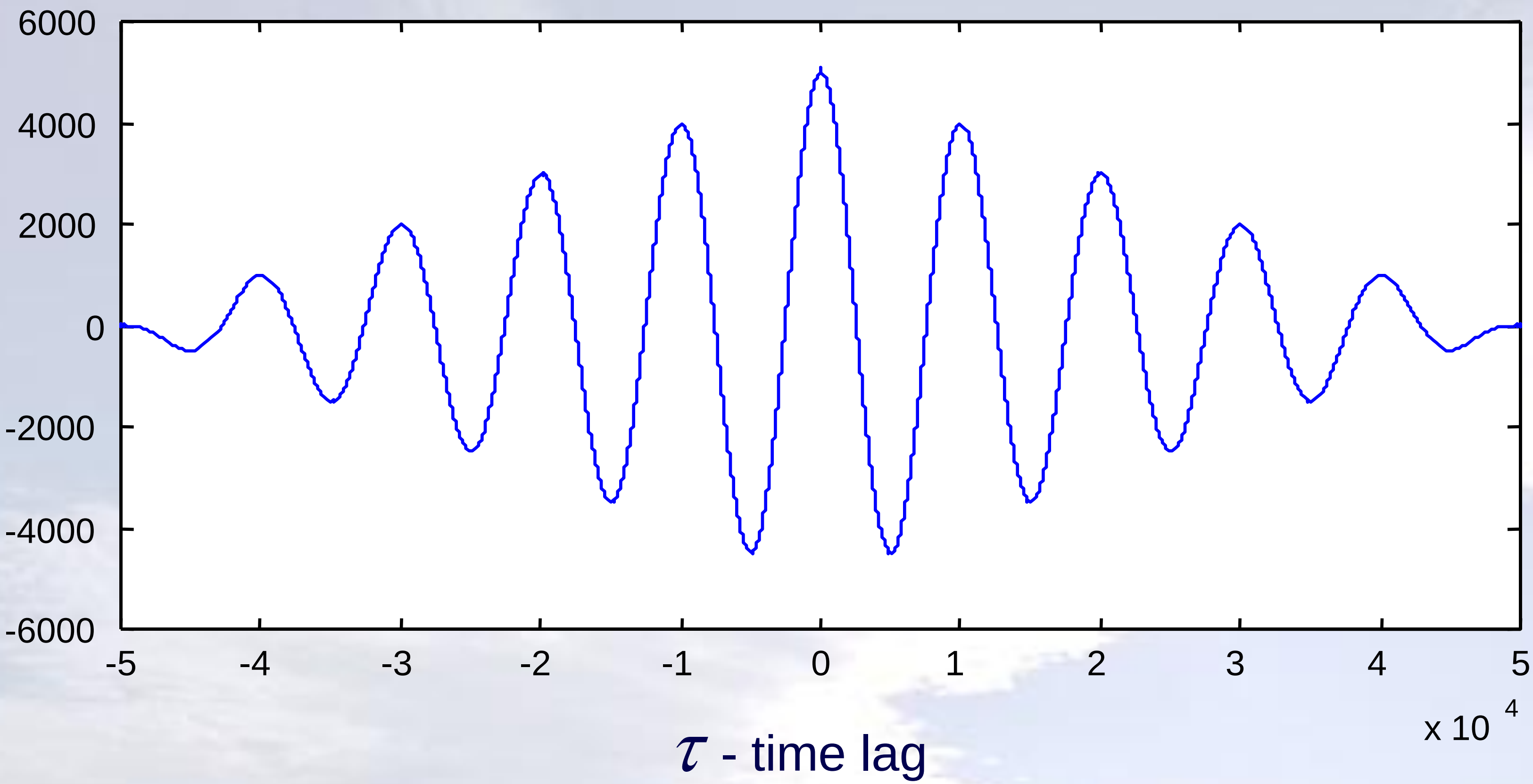
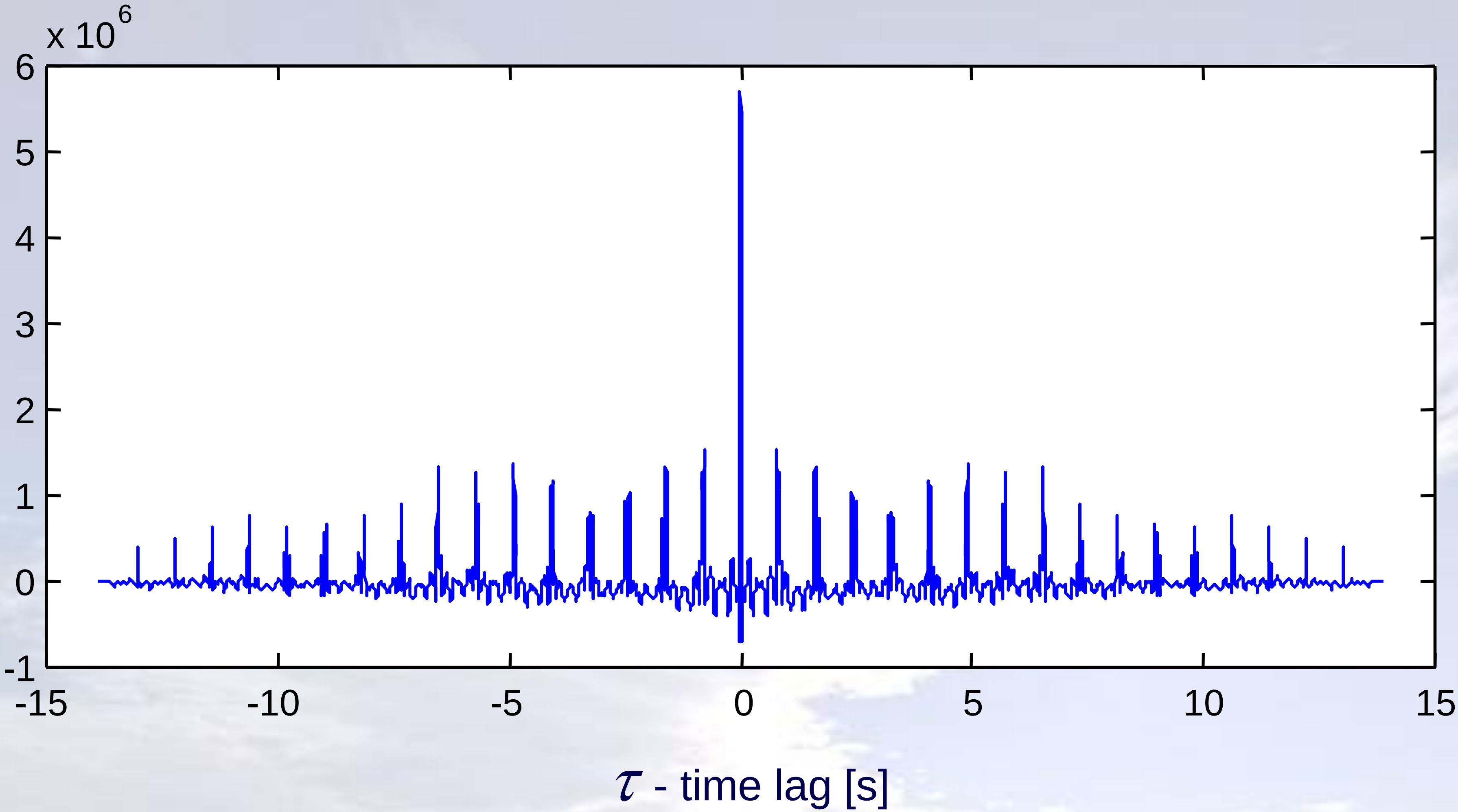
Autocorrelation function of harmonic signal



Autocorrelation function – open questions

1. Compare the length of the signal with the lenght of autocorrelation function.
2. What properties of autocorrelation can you notice?
3. What can be the application of autocorrelation function?

ECG signal autocorrelation



Cross-correlation function

Continuous signals:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt$$

Discrete signals:

```
#Python
R=correlate(x,y,mode='full')
```

$$R_{xy}(m) = \sum_{n=-\infty}^{n=\infty} x(n)\,y(n+m)$$

Don’t forget to remove the mean from the signals!

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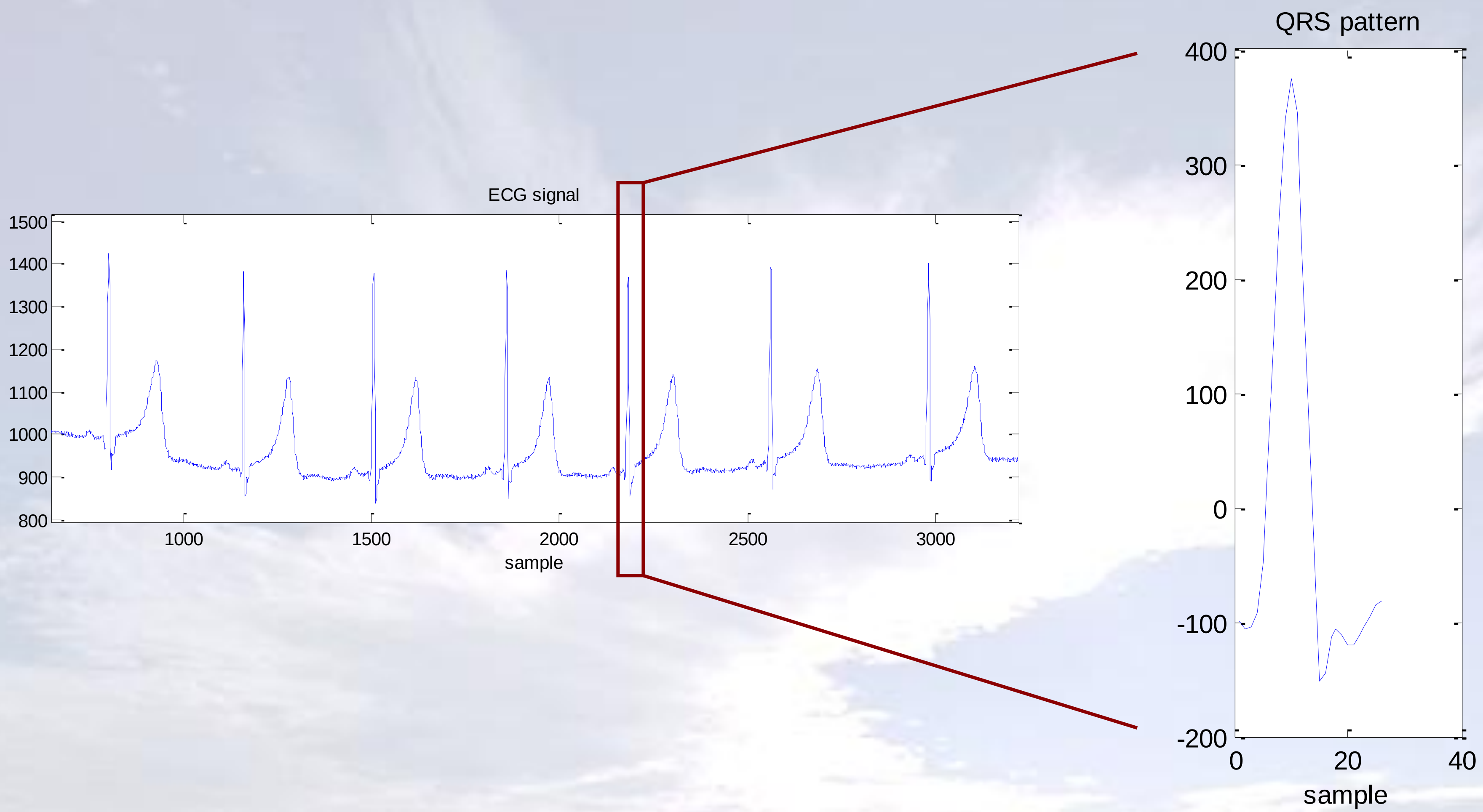
Cross-correlation function – example1

QRS detection by correlation with QRS pattern

1. Determine the QRS pattern of the sample ECG signal.
2. Use the pattern to detect other QRS comlexes
3. Keep the results in order to compare them with the other detection methods

Cross-correlation function – example1



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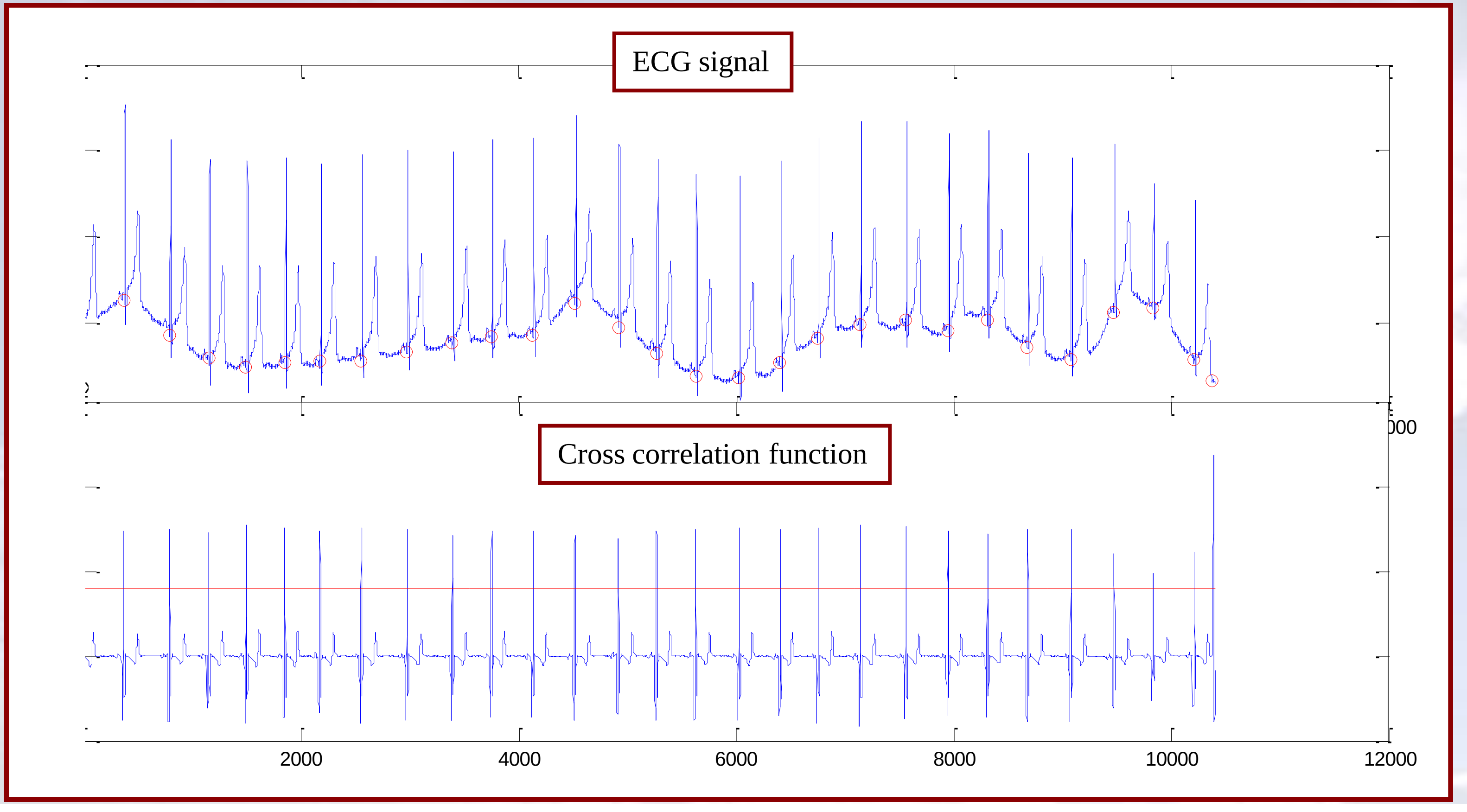
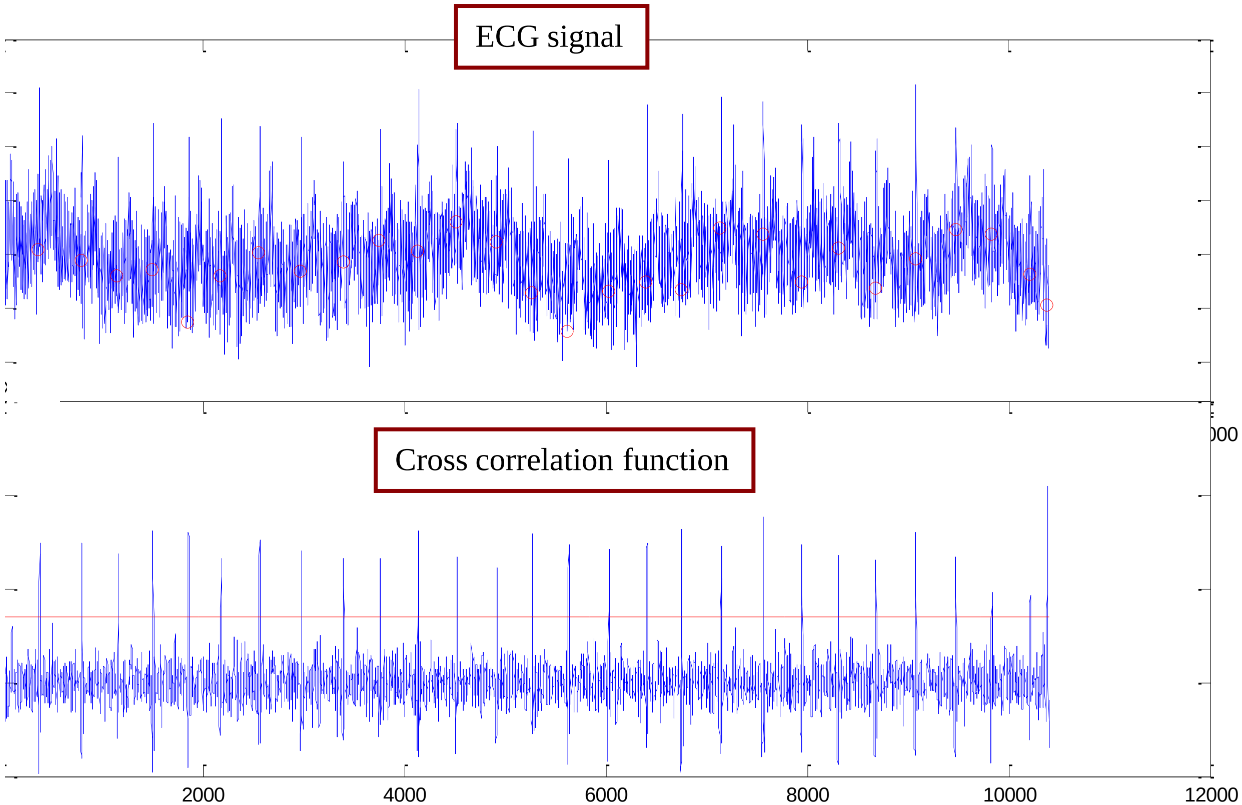
Cross-correlation function – example1

kor=xcorr(ECGsignal,QRSpattern);



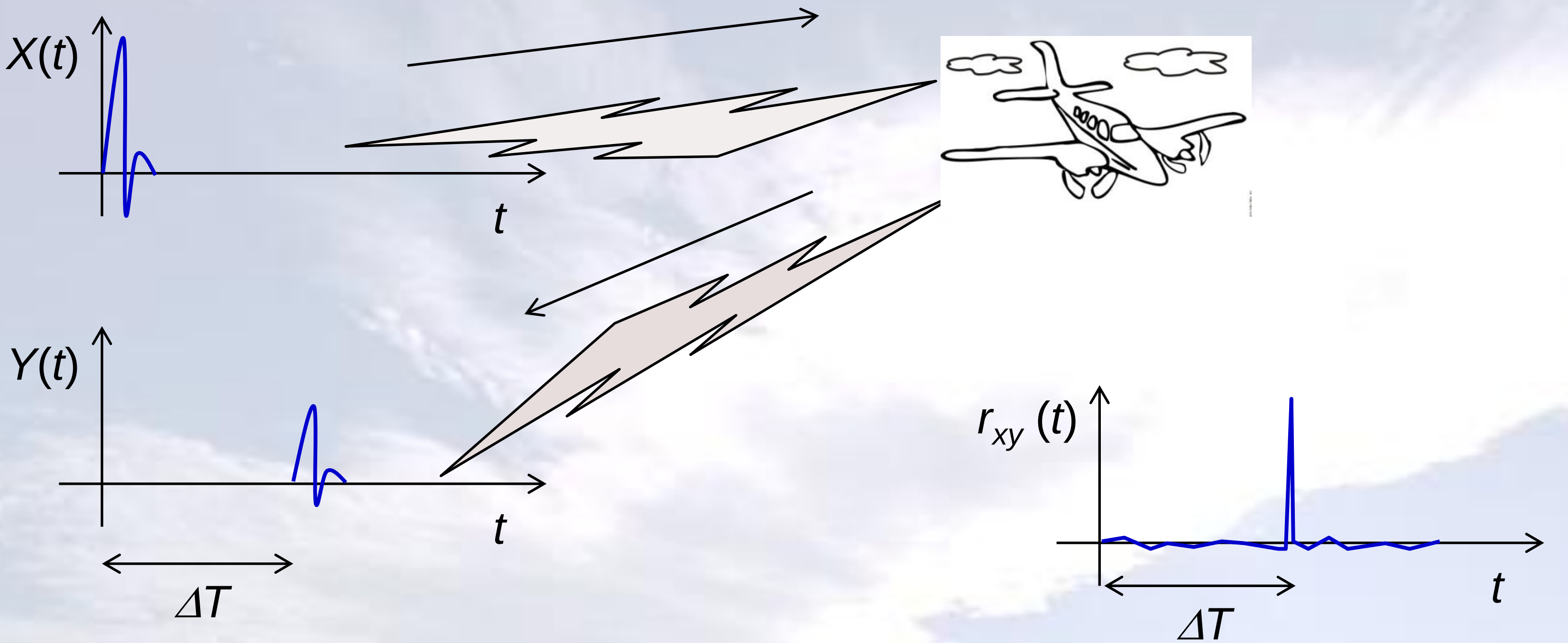
Cross-correlation function – example1

```
kor=xcorr(ECGsignal,QRSpattern)
```



Cross-correlation function – example2

Delay estimation of radar signals:

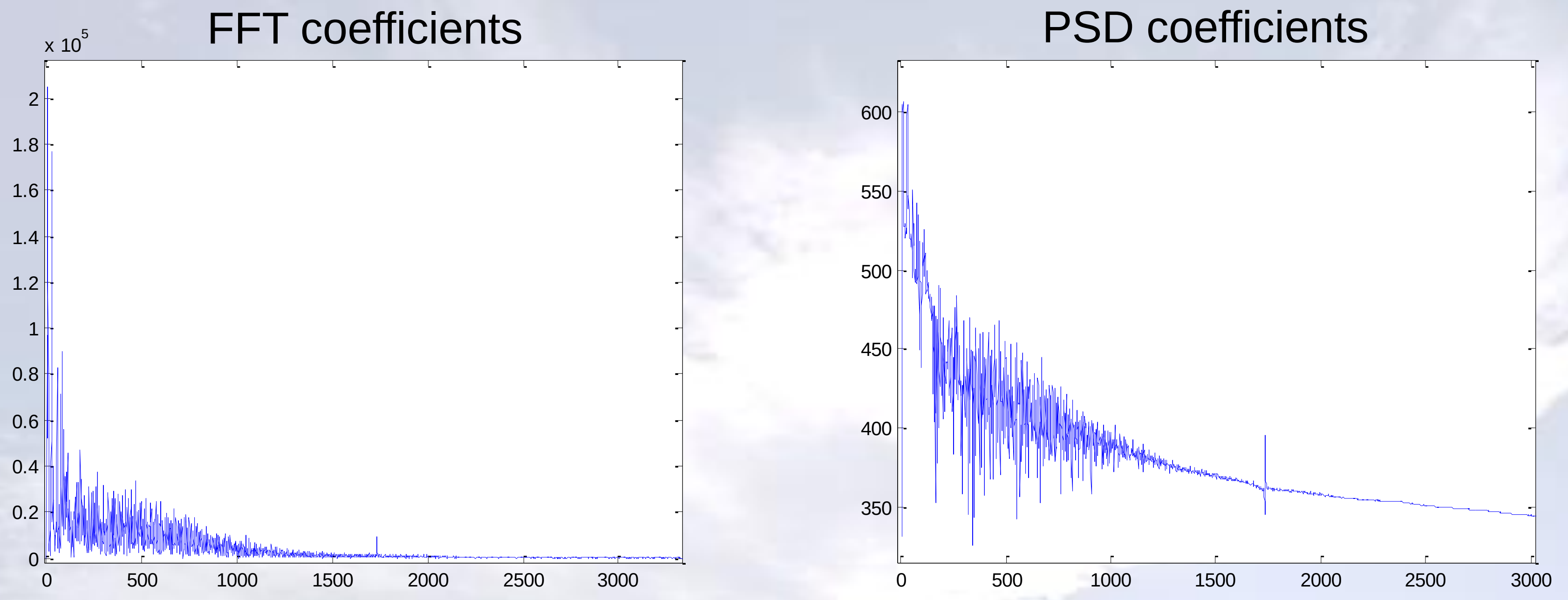


Power Spectral Density - PSD

- **Autocorrelation function** of a random signal is deterministic => its Fourier spectrum may be computed

$$FT\left\{R_{xx}(\tau)\right\} = X(j\omega)X^*(j\omega) = \left|X(j\omega)\right|^2$$

Power Spectral Density - PSD



Correlation coefficients

Similarity of two signals may be assessed with correlation coefficients.

covariance

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x \sigma_y}} = \frac{\sum_{n=1}^{N}[x(n) - x][y(n) - y]}{\sqrt{\sum_{n=1}^{N}[x(n) - x]^2 \sum_{n=1}^{N}[y(n) - y]^2}}$$

$$-1 \le r_{xy} \le 1$$

```
#Python

corrcoef?
```

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Correlation coefficient - example

Compute correlation coefficient between signals *a* and *b*:

$$a = [\ \ 1, 2, 3]$$
$$b = [-3, 4, 2]$$

$$\Rightarrow \quad r_{xy} \cong 0.6934$$

$$r_{xy} = \frac{\sum_{n=1}^{N} [x(n) - \bar{x}][y(n) - \bar{y}]}{\sqrt{\sum_{n=1}^{N} [x(n) - \bar{x}]^2 \sum_{n=1}^{N} [y(n) - \bar{y}]^2}}$$

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Correlation coefficients

EXERCISE: Create 3 vectors:

1. Vector x_1 whose elements are heights of the members of the group in meters
2. Vector x_2 – the weights in the same order as in x_1 (do not cheat!)
3. Vector x_3 – shoe size

Determine correlation coefficients of the vectors and point the most correlated pair of vectors.

Probability distribution functions

Uniform:

$$p(x)=\begin{cases}\dfrac{1}{x_1-x_0} & for \quad x_0 \le x \le x_1 \\ 0 & for\ other\ \ x\end{cases}$$

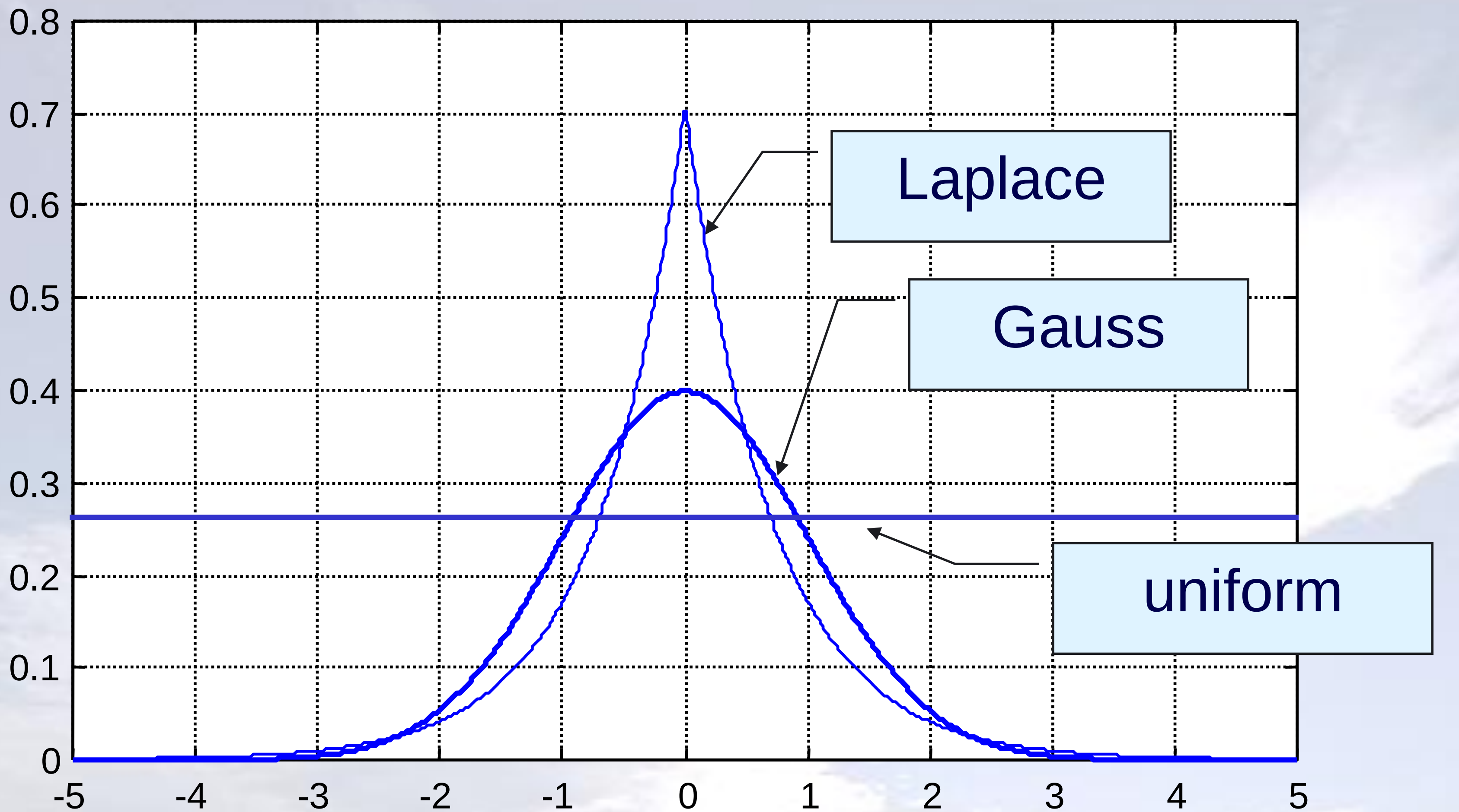
Laplace:

$$p(x)=\frac{1}{\sigma\sqrt{2}}e^{-\frac{\sqrt{2}\left|x-\mu\right|}{\sigma}}$$

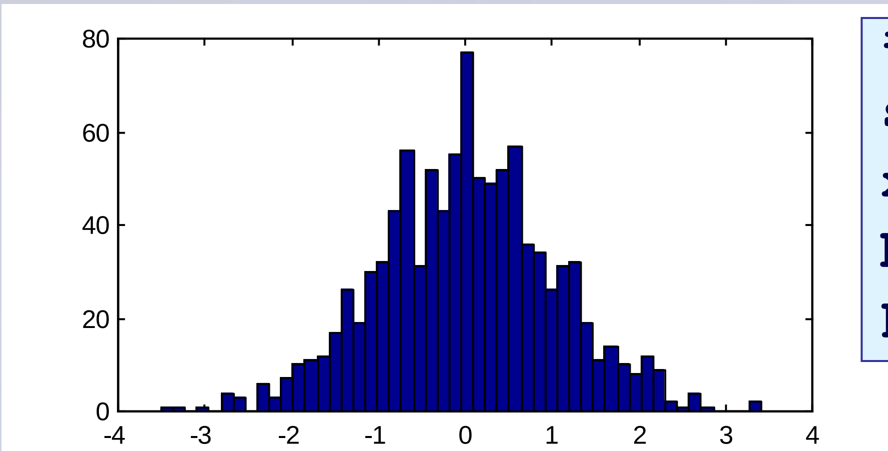
Normal (Gauss):

$$p(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

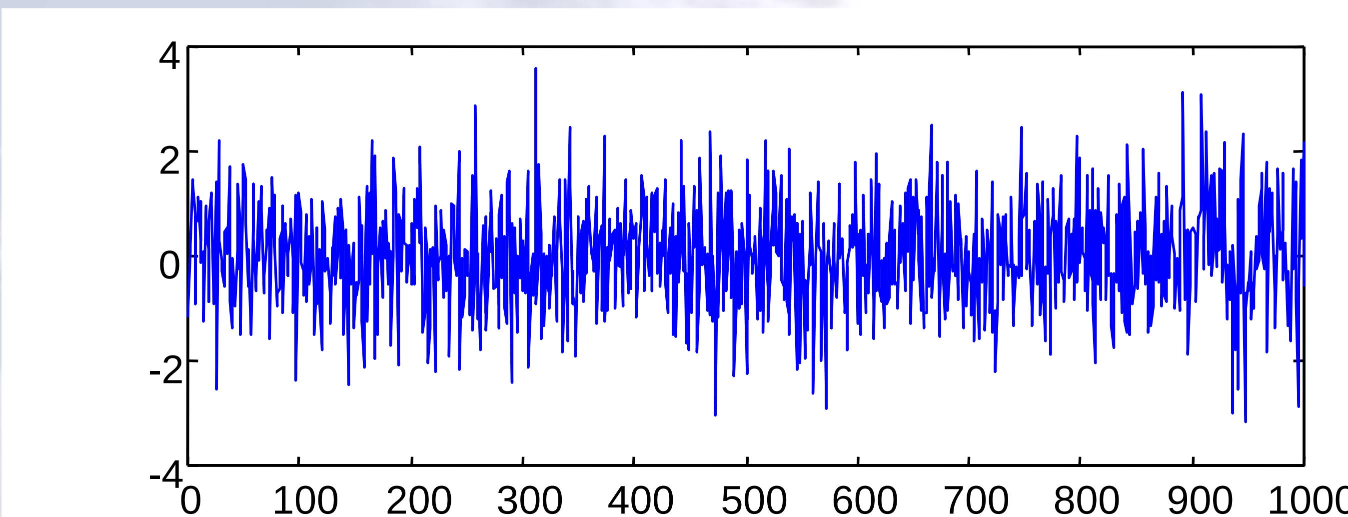
Probability distribution functions



Gaussian noise

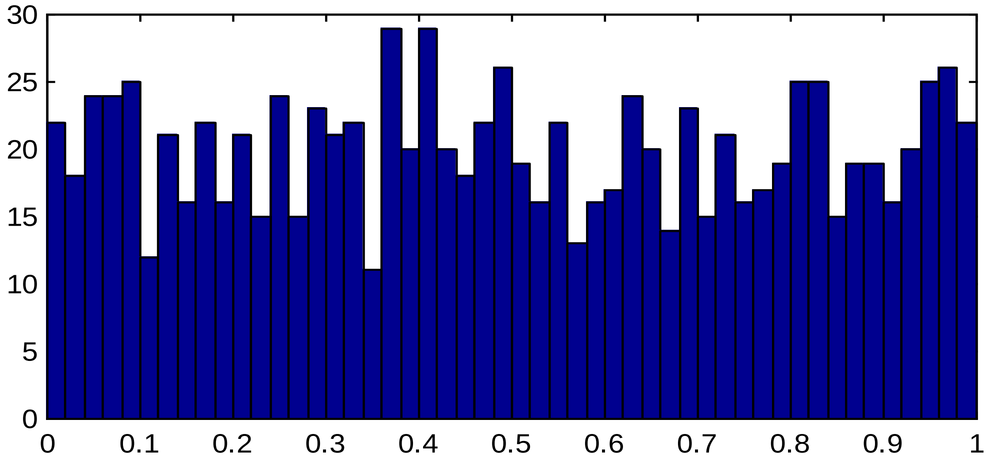



```
#Python
sigma=1
x=sigma*random.randn(1000)
bins=linspace(-5,5,50)
hist(x,bins, normed=‘True’)
```

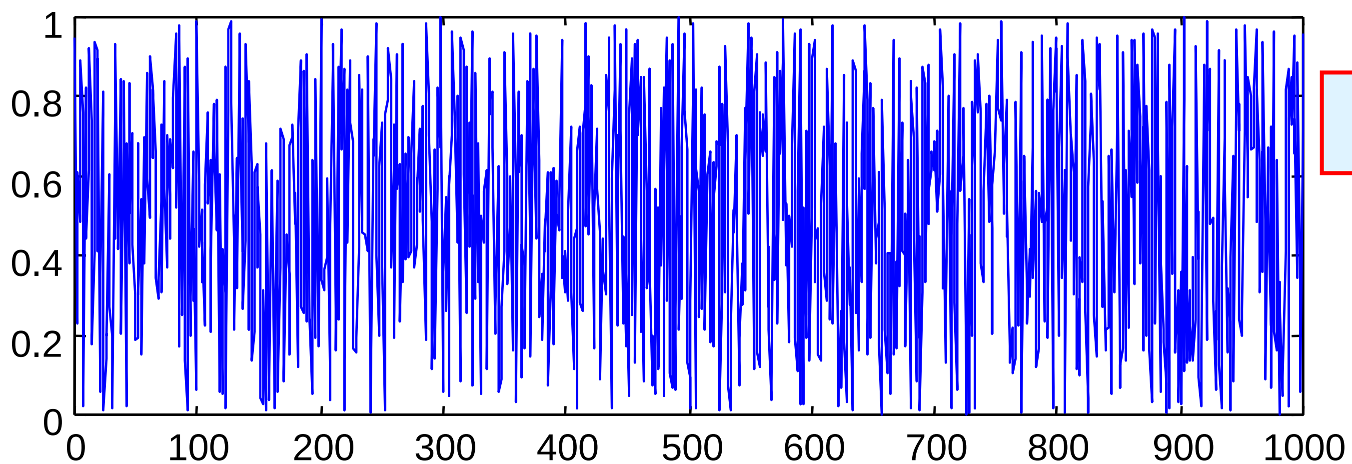


$$x = 0,\ \sigma = 1$$

Noise with uniform distribution



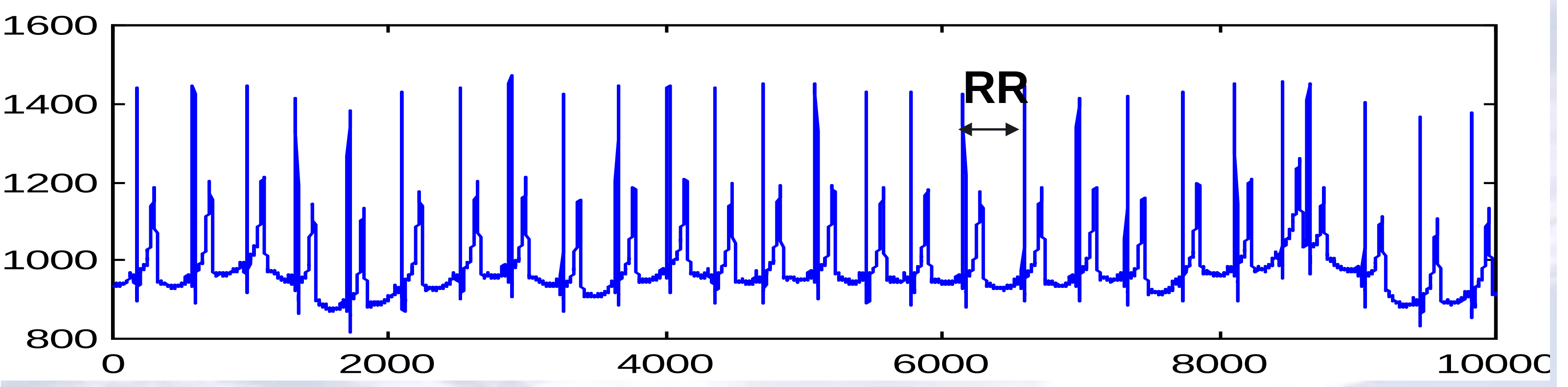

```
#Python
range=1
y=range*random.rand(1000)
bins=linspace(-1,1.5,50)
hist(y,bins, normed='True')
```



$$\boxed{x = 0.5\,!}$$

Markov Process

$$p\left[x(n)/x(n-1),x(n-2),\ldots,x(1)\right]=p\left[x(n)/x(n-1)\right]$$



Is the series $\{RR_1, RR_2, \ldots, RR_n, \ldots\}$ a Markov process?

Poisson discrete distribution

Poisson distribution is applied in observations of independent phenomena with low probability of success (eg radioactive decay)

Probability of *j* events in time Δt:

$$p\left[Z(\Delta t) = j\right] = \frac{(\lambda \Delta t)^j}{j!} e^{-\lambda \Delta t}, \quad \lambda > 0 \Rightarrow$$

$$E(Z) = \lambda$$

$$D^2(Z) = \lambda$$

where λ is a mean number of events in Δt.

Application:

RR intervals modeling**,** EMG signal, neural impulses, quality control

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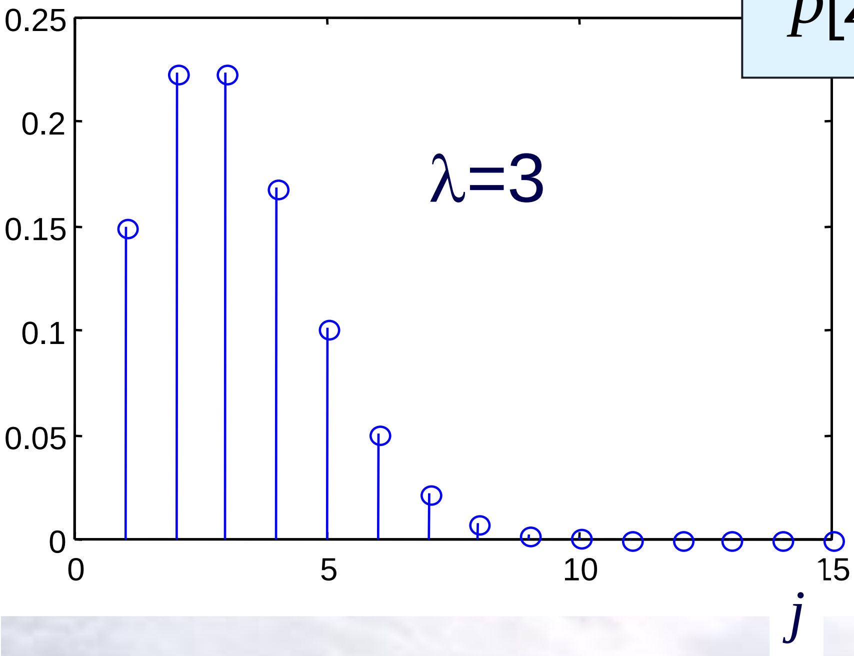
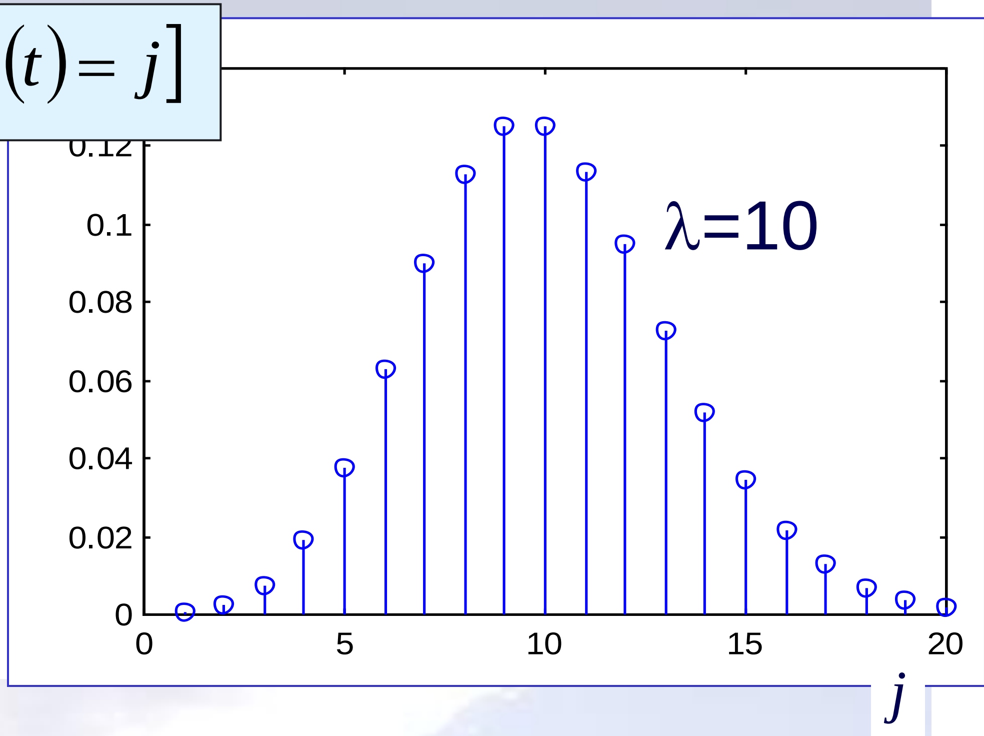
Poisson distribution

$$p[Z(t) = j]$$





Poisson process describes so called memoryless process, where the number of events in time is independent on the past.

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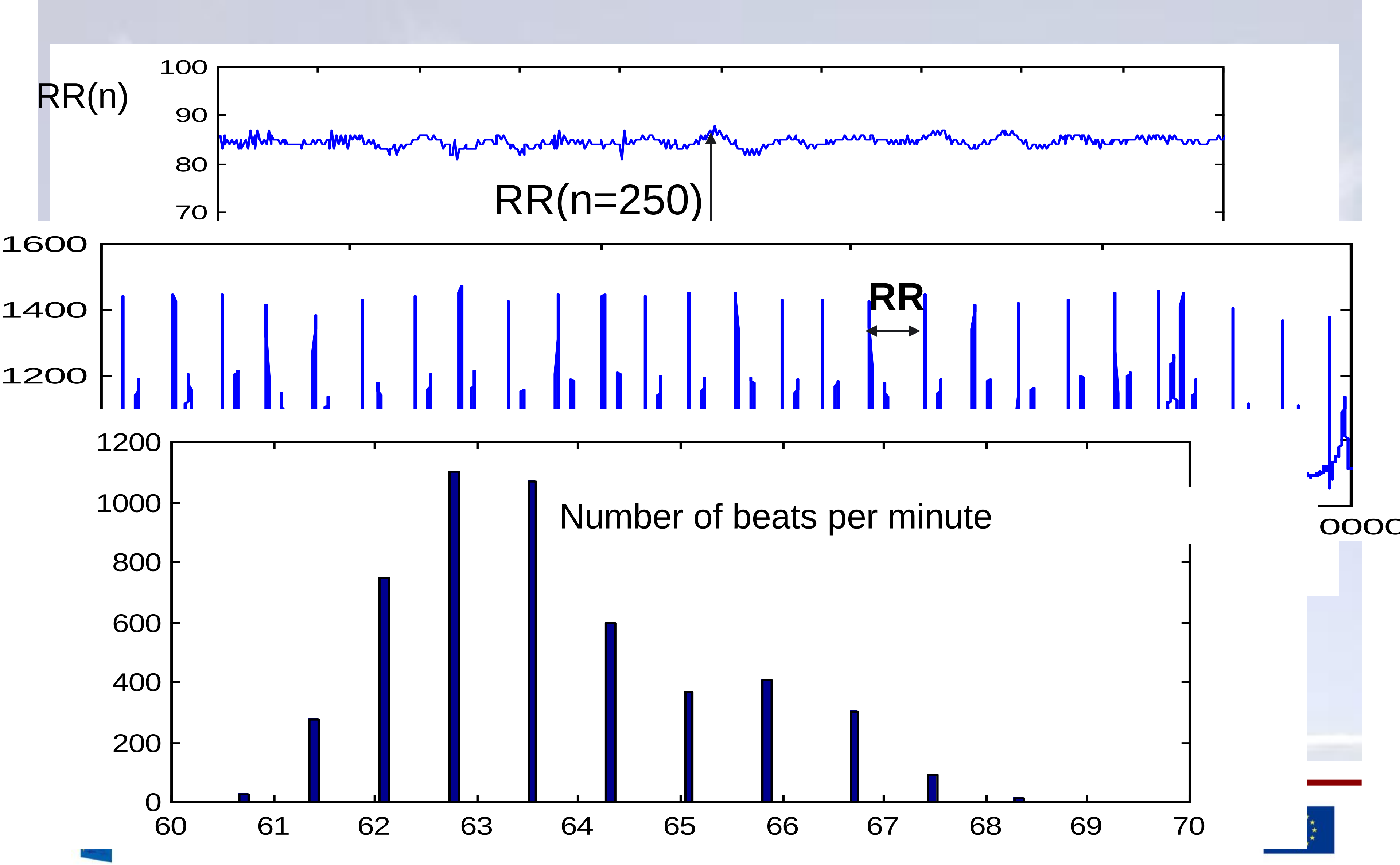
Heart rate variability - HRV



Linear regression

Let's consider:

$$y = \alpha_0 + \alpha_1 x + \varepsilon$$

where:

- x – independent variable,
- y – dependent variable,
- ε - noise (eg. measurement error),
- α_0, α_1 – regression coefficients.

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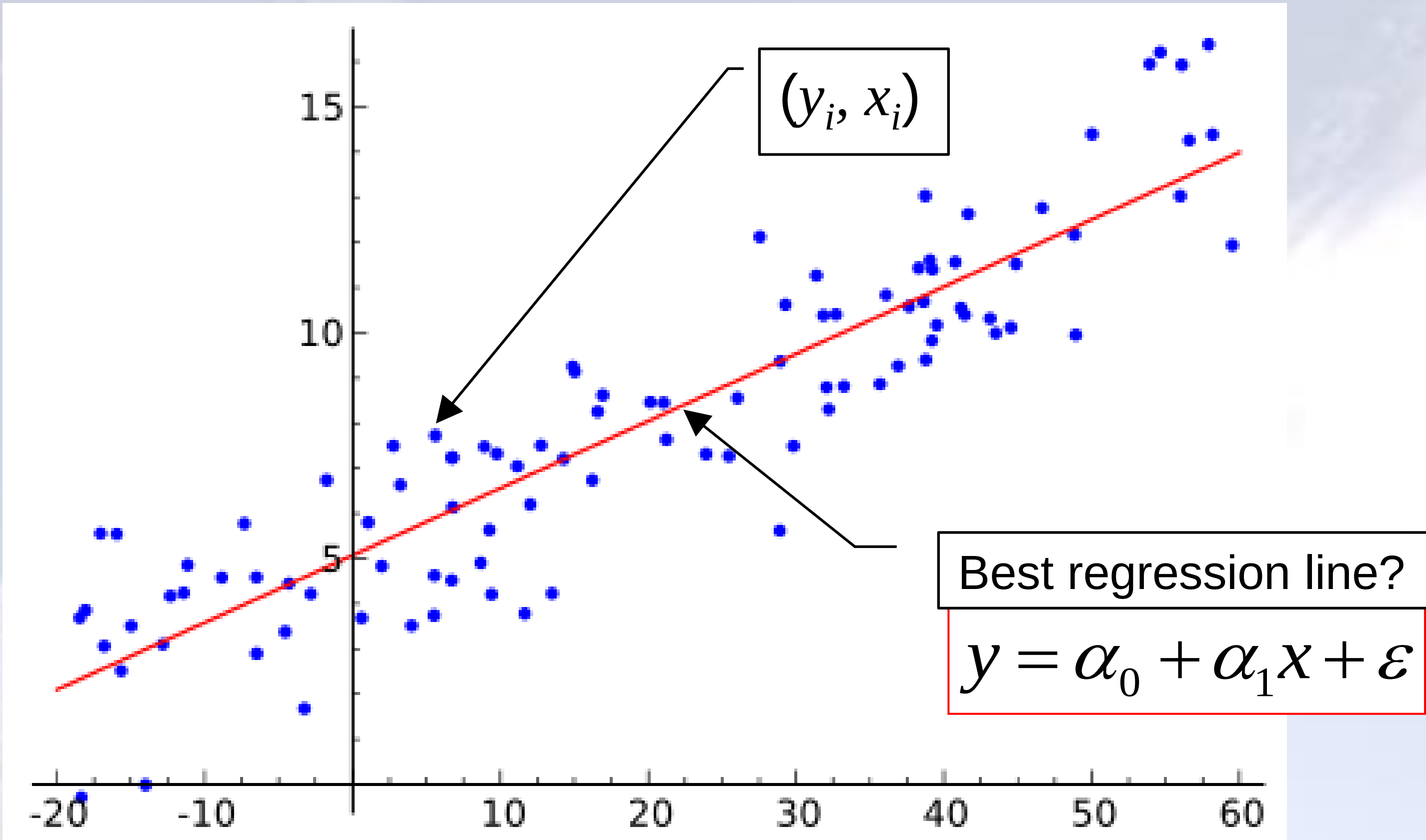
Linear regression



$$y = \alpha_0 + \alpha_1 x + \varepsilon$$

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Linear regression - estimation

Aim:

Suppose P observations (x_1, y_1), (x_2, y_2), ..., (x_P, y_P) are given. Determine α_0, α_1, which most accurately relate variables x i y.

Solution:

Estimates of the coefficients may be derived by means of the least square method.

$$\varepsilon_i = \underbrace{y_i}_{\text{observation}} - \underbrace{(\alpha_0 + \alpha_1 x)}_{\text{Linear regression model}} = y_i - \hat{y}_i$$

Linear regression - estimation

summed squared error - SSE

$$SSE = \sum_{i=1}^{P} \varepsilon_i^2 = \sum_{i=1}^{P} \left(y_i - (\hat{\alpha}_0 + \hat{\alpha}_1 x_i) \right)^2$$

According to the idea of least squares method proposed by Gauss in XIX-th century.

The mininum of the function is to be established.

Linear regression - estimation

The minimum is determined by setting the partial derivatives to zero.

$$\begin{cases} \dfrac{\partial(SEE)}{\partial\hat{\alpha}_0} = \dfrac{\partial}{\partial\hat{\alpha}_0}\left(\sum_{i=1}^{P}\left(y_i - \hat{\alpha}_0 - \hat{\alpha}_1 x_i\right)^2\right) = 0 \\ \dfrac{\partial(SEE)}{\partial\hat{\alpha}_1} = \dfrac{\partial}{\partial\hat{\alpha}_1}\left(\sum_{i=1}^{P}\left(y_i - \hat{\alpha}_0 - \hat{\alpha}_1 x_i\right)^2\right) = 0 \end{cases}$$

So called normal set of equations

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Linear regression - estimation

The solution to the normal set of equations:

$$\begin{cases} \boxed{\hat{\alpha}_0 = \bar{y} - \hat{\alpha}_1 \bar{x}} \\ \boxed{\hat{\alpha}_1 = \dfrac{\sum_{i=1}^{P} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{P} (x_i - \bar{x})^2}} \end{cases}$$

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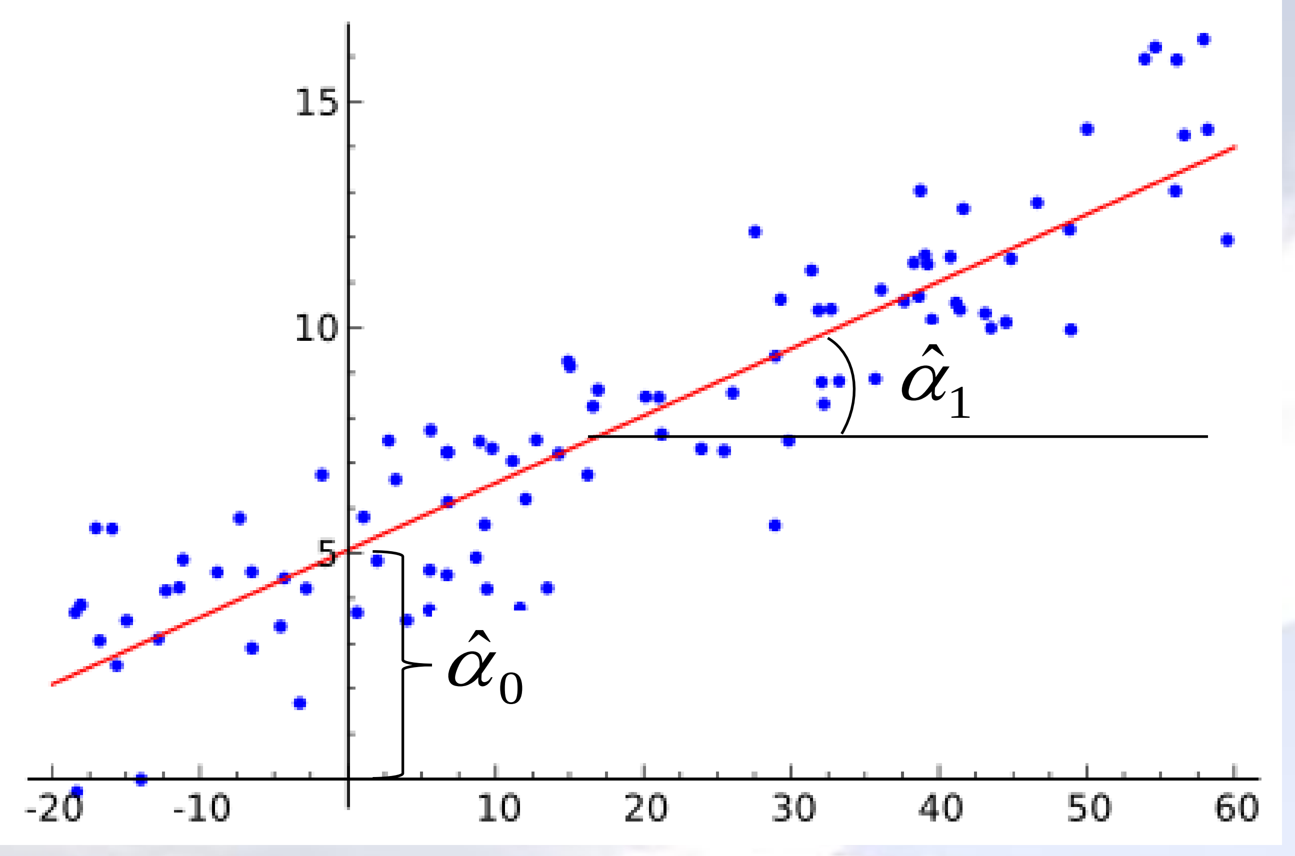
Linear regression - estimation



Source: Wikipedia

Linear regression

Higher order regression:

$$y_i = \alpha_0 + \alpha_1 x_{i1} + \alpha_1 x_{i1} + \ldots + \alpha_N x_{iN}, \quad i = 1, \ldots, P$$

$$y_i = \alpha_0 + \sum_{j=1}^{N} \alpha_j x_{ij}, \quad i = 1, \ldots, P$$

Analogous solution:

$$\frac{\partial (SEE)}{\partial \hat{\alpha}_j} = 0, \quad j = 0, \ldots N$$

(*N*+1 equations with *N*+1 variables)

Other regression models

Polynomial regression:

$$y_i = \alpha_0 + \alpha_1 x_{i1}^1 + \alpha_1 x_{i2}^2 + \ldots + \alpha_N x_{iN}^N, \quad i = 1, \ldots P$$

Nonlinear regression:

$$y_i = \alpha_0 e^{\alpha_1 x_i}$$

$$y_i = \alpha_0 \alpha_1^{x_i}$$

$$y_i = \frac{\alpha_0}{1 + \alpha_1 e^{x_i}}$$

Trajectory reconstruction

Takens (1981) proved, the properties of the trajectories of the dynamic system may be reconstructed from the samples of a 1-dimensional registration of the activity of the given system.

$$\boldsymbol{y}(k)=\left[y(k),y(k-\Delta t),\ldots,y(k-(D-1)\Delta t)\right]^T$$

For the appropriate length D of the vector ***y***(*k*) (*time delay embedding*)

Trajectory reconstruction

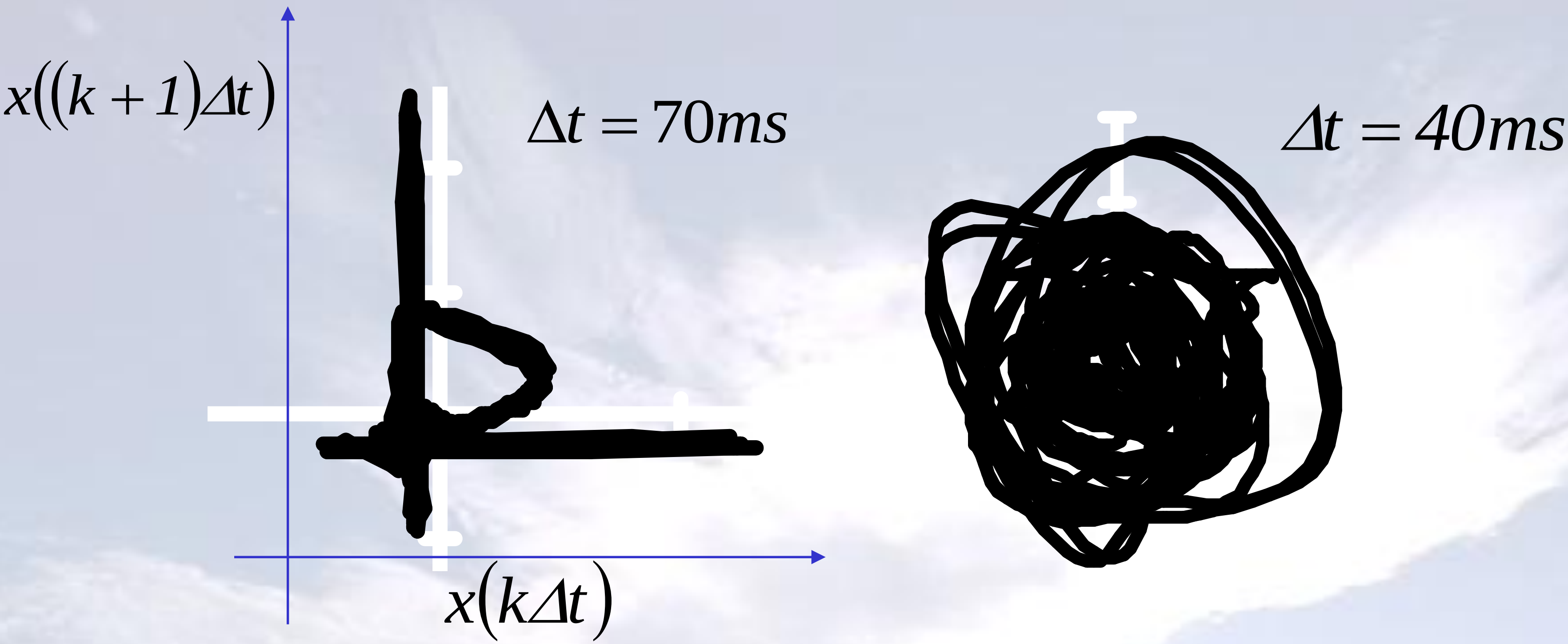
In practice Takens theorem means that there exists a dependance:

?

$$y\left(\left(k+1\right)\Delta t\right)=F\left(\mathbf{y}\left(k\right)\right)$$

due to which it is possible to forecast the sample $y(k+1)\Delta t$ of the signal reflecting the behaviour of the given system.

Trajectory reconstruction for heart rhythm



Normal heart rhythm

Ventricle fibrillation

Δt – first zero of the autocorrelation function of the series $y(k)$

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Deterministic chaos

Consider the simple dynamic system:

$$x(n+1) = ax(n)(1 - x(n))$$

Simulating the change of number of population x in limited environment. This number is normalized to $\langle 0,1 \rangle$, therefore the value of parameter a belongs to $\langle 0,4 \rangle$.

This is the, so called, **logistic equation**.

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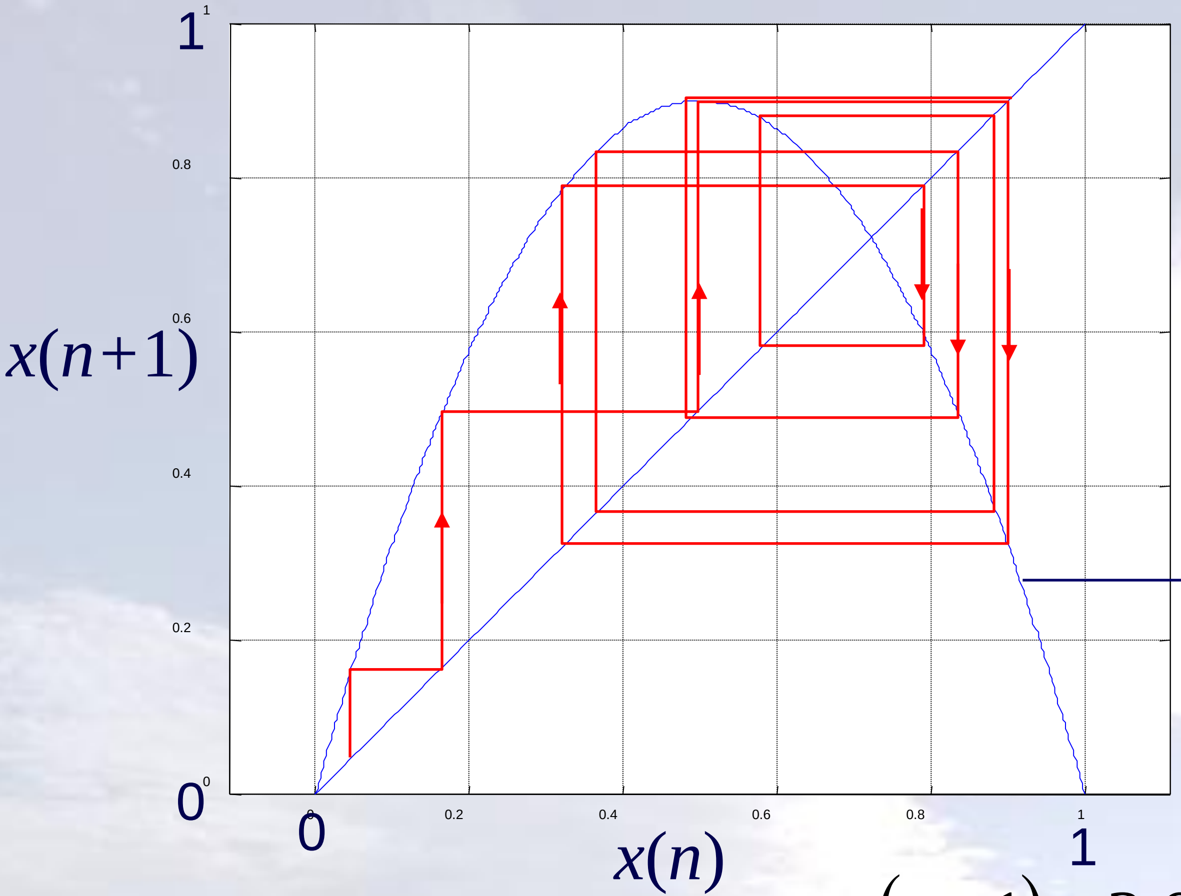
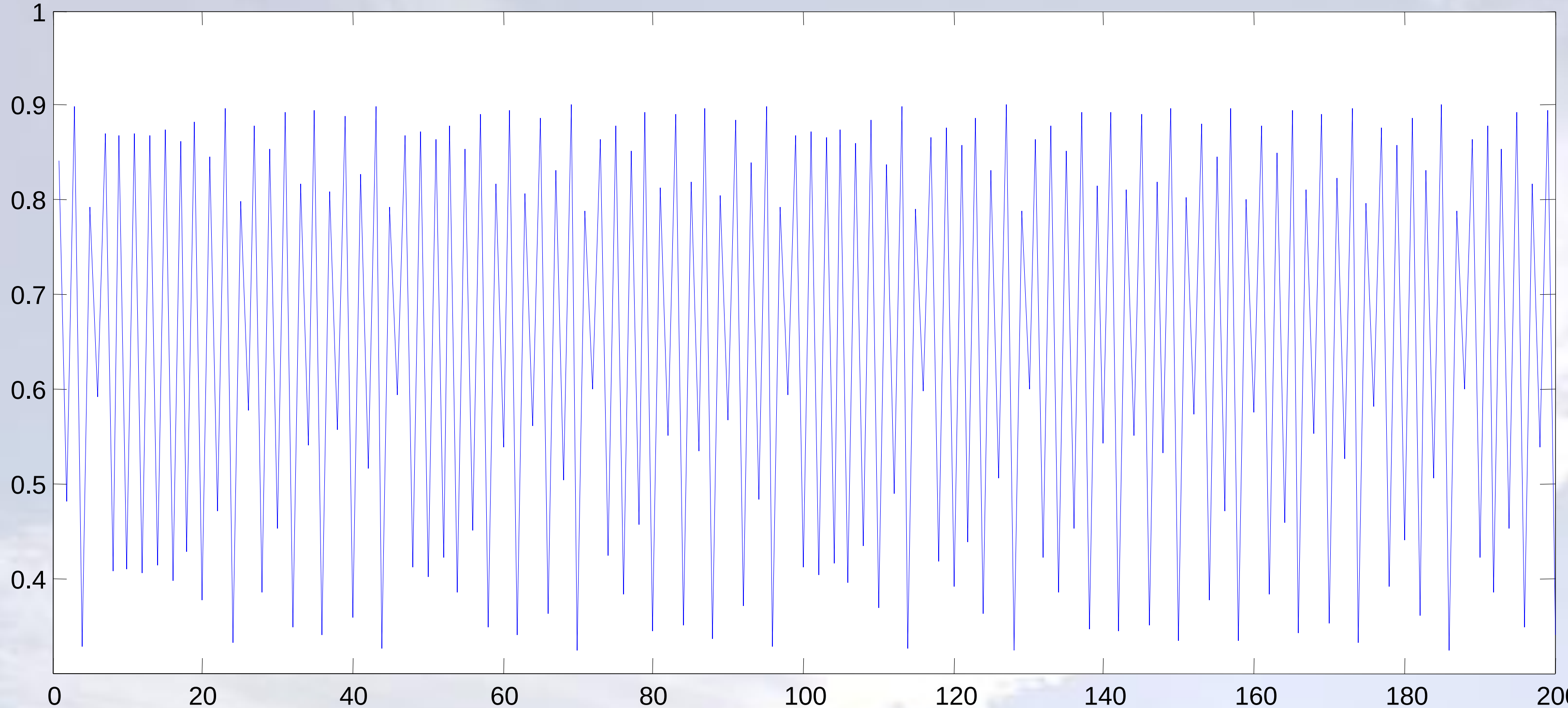
Deterministic chaos



So called „square projection”

$$x(n+1) = 3.6x(n)(1 - x(n))$$

Deterministically chaotic process



Deterministic chaos

The properties of the deterministic, chaotic processes:

- may be generated by simple dynamic systems *,
- the dynamics has „unpredictable” random character (so called strage attractors) → demo Matlab
- Dynamics highly dependent on the initial conditions („butterfly effect”)

*) R, May: „ … *life would be easier if it was taken into account, that simple dynamic systems not always lead to simple dynamic evolution*”

Analysis of random signals - summary

1. Deterministic vs stochastic signals
2. Mean, variance, standard deviation
3. Autocorrelation function
4. Cross-correlation function
5. Correlation coefficient
6. Power Spectral Density
7. Distributions
8. System dynamics

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„SIGNAL PROCESSING”

Prezentacja multimedialna współfinansowana przez Unię Europejską w ramach Europejskiego Funduszu Społecznego w projekcie pt.

„Innowacyjna dydaktyka bez ograniczeń - zintegrowany rozwój Politechniki Łódzkiej - zarządzanie Uczelnią, nowoczesna oferta edukacyjna i wzmacniania zdolności do zatrudniania osób niepełnosprawnych”